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ROMAN DOMINATION IN GRAPHS: THE CLASS $\mathcal{R}_{\mathbf{UVR}}$

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Abstract. For a graph $G = (V, E)$, a Roman dominating function $f : V \to \{0, 1, 2\}$ has the property that every vertex $v \in V$ with $f(v) = 0$ has a neighbor u with $f(u) = 2$. The weight of a Roman dominating function f is the sum $f(V) = \cup_{v\in V} f(v)$, and the minimum weight of a Roman dominating function on G is the Roman domination number $\gamma_R(G)$ of G. The Roman bondage number $b_R(G)$ of G is the minimum cardinality of all sets $F \subseteq E$ for which $\gamma_R(G - F) > \gamma_R(G)$. A graph G is in the class $\mathcal{R}_{UVR}$ if the Roman domination number remains unchanged when a vertex is deleted. In this paper we obtain tight upper bounds for $\gamma_R(G)$ and $b_R(G)$ provided a graph G is in $\mathcal{R}_{UVR}$. We present necessary and sufficient conditions for a tree to be in the class $\mathcal{R}_{UVR}$. We give a constructive characterization of $\mathcal{R}_{UVR}$-trees using labellings.

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1. Introduction

All graphs considered in this paper are finite, undirected, loopless, and without multiple edges. We refer the reader to the book [25] for graph theory notation and terminology not described in this paper. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. We write K_n for the *complete graph* of order n and C_n for a *cycle* of length n. Let P_m denote the *path* with m vertices. In a graph G, for a subset $S \subseteq V(G)$ the *subgraph induced* by S is the graph $\langle S\rangle$ with vertex set S and edge set $\{xy \in E(G) \colon x, y \in S\}$. The *complement* $\overline{G}$ of G is the simple graph whose vertex set is $V(G)$ and whose edges are the pairs of nonadjacent vertices of G. The *join* of simple graphs G and H, written $G \vee H$, is the graph obtained from the disjoint union of G and H by adding the edges $\{xy \mid x \in V(G),\ y \in V(H)\}$. For any vertex x of a graph G, $N_G(x)$ denotes the set of all neighbors of x in G, $N_G[x] = N_G(x) \cup \{x\}$ and the *degree* of x is $deg_G(x) = |N_G(x)|$. The *minimum* and *maximum* degree of a graph G are denoted by

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$\delta(G)$ and $\Delta(G)$, respectively. For a graph G, let $x \in X \subseteq V(G)$. A vertex $y \in V(G)$ is a *X-private neighbor* of x if $N_G[y] \cap X = \{x\}$. The *X-private neighborhood* of x, denoted $pn_G[x, X]$, is the set of all X-private neighbors of x.

A set $D \subseteq V(G)$ is a *dominating set* if for each $x \in V(G)$ either $v \in D$ or x is adjacent to some $y \in D$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G, and a dominating set D of minimum cardinality is called a *γ-set* of G. If D is a γ-set of G, then $pn[v, S] \neq \emptyset$ for each $v \in D$. An efficient dominating set in a graph G is a set $S \subseteq V(G)$ such that $\{N[s] \mid s \in S\}$ is a partition of $V(G)$. All efficient dominating sets in graph G have the same cardinality that is equal to $\gamma(G)$ ([10]). The concept of domination in graphs has many applications to several fields. Domination naturally arises in facility location problems, in problems involving finding sets of representatives, in monitoring communication or electrical networks, and in land surveying. Many variants of the basic concepts of domination have appeared in the literature. We refer to [10, 11] for a survey of the area.

A variation of domination called Roman domination was introduced independently by Arquilla and Fredricksen [1], ReVelle [19, 20] and Stewart [24], which was motivated with the following legend. In the 4th century A.D., Emperor Constantine the Great issued a decree to ensure the protection of the Roman empire. Constantine ordered that each city in the empire either has a legion stationed within it for defense or lies near a city with two standing legions. This way, if a defenseless city were attacked, a nearby city could dispatch reinforcements without leaving itself defenseless. The natural problem is to determine how few legions suffice to protect the empire. The concept of Roman domination can be formulated in terms of graphs. More formally, following Cockayne et al. [6], a *Roman dominating function* (*RDF*) on a graph G is a vertex labeling $f : V(G) \to \{0, 1, 2\}$ such that every vertex with label 0 has a neighbor with label 2. For an RDF f, let

$V_i^f = \{v \in V(G) : f(v) = i\}$ for i = 0, 1, 2. Since this partition determines f, we can equivalently write $f = (V_0^f; V_1^f; V_2^f)$. The weight $f(V(G))$ of an RDF f on G is the value $\Sigma_{v \in V(G)} f(v)$, which equals $|V_1^f| + 2|V_2^f|$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of a Roman dominating function on G. Thus, $\gamma_R(G)$ is the minimum number of legions needed to protect cities whose adjacency graph is G. A function $f = (V_0^f; V_1^f; V_2^f)$ is called a γ_R-*function* on G, if it is a Roman dominating function and $f(V(G)) = \gamma_R(G)$. If f is an RDF on a graph G and H is a subgraph of G, then we denote the restriction of f on H by $f|_H$.

Cockayne et al.[6] showed that

$$\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G). \tag{1}$$

A graph G is called to be *Roman* if $\gamma_{\mathrm{R}}(G) = 2\gamma(G)$. All Roman paths and cycles are P_{3k}, C_{3k}, P_{3k+2}, and C_{3k+2} ([6]). Liedloff et al. [12] and Liu and Chang [14] investigated algorithmic aspect of Roman domination. Applications of Roman domination were also shown in [4]. Also see ReVelle and Rosing [21] for an integer programming formulation of the problem.

The concept of Roman bondage in graphs was introduced by Jafari Rad and Volkmann in [16]. Let G be a graph with maximum degree at least two. The Roman bondage number $b_R(G)$ of G is the minimum cardinality of all sets $E_1 \subseteq E(G)$ for which $\gamma_R(G - E_1) > \gamma_R(G)$. This number is a measure of the efficiency of Roman domination in graphs. In [2], Bahremandpour et al. showed that the decision problem for $b_R(G)$ is NP-hard even for bipartite graphs. For more information we refer the reader to [2, 16, 22, 23].

When we remove a vertex from a graph G, the Roman domination number can increase or decrease, or remain the same. The following classes of graphs were defined and first investigated in [18] by Jafari Rad and Volkmann.

- $\mathcal{R}_{CVR}$ is the class of graphs G such that $\gamma_R(G-v) \neq \gamma_R(G)$ for all $v \in V(G)$,
- $\mathcal{R}_{UVR}$ is the class of graphs G such that $\gamma_R(G-v) = \gamma_R(G)$ for all $v \in V(G)$.

Here we concentrate on the class $\mathcal{R}_{UVR}$. The paper is organized as follows. Section 2 contains some known facts about Roman domination in graphs. In Section 3 we obtain tight upper bounds for $\gamma_R(G)$ and $b_R(G)$ provided a graph G is in $\mathcal{R}_{UVR}$. In Section 4 we present necessary and sufficient conditions for a tree to be in the class $\mathcal{R}_{UVR}$. We also give a constructive characterization of $\mathcal{R}_{UVR}$-trees using labellings.

2. Some known results

Lemma A. ([6]) *Let $f = (V_0; V_1; V_2)$ be any γ_R-function on a graph G. Then each component of a graph $\langle V_1 \rangle$ has order at least 2 and no edge of G join V_1 and V_2.*

Lemma A will be used in the sequel without specific reference.

Lemma B. ([17]) *Let v be a vertex of a graph G. Then $\gamma_R(G - v) < \gamma_R(G)$ if and only if there is a γ_R-function $f = (V_0; V_1; V_2)$ on G such that $v \in V_1$. If $\gamma_R(G - v) < \gamma_R(G)$ then $\gamma_R(G - v) = \gamma_R(G) - 1$.*

Lemma C. ([18]) *If e is an edge of a graph G, then $\gamma_R(G - e) \geq \gamma_R(G)$.*

Theorem D. ([6]) *A graph G is Roman if and only if it has a γ_R-function f with $V_1^f = \emptyset$.*

Theorem E. [8] *Let T be a tree of order at least 3 and let D be a dominating set of G. Then D is the unique γ-set of T if and only if every vertex in D has at least two nonadjacent D-private neighbors.*

The differential of a graph was introduced in [15] in 2006, motivated by its applications to information diffusion in social networks. The *differential of a vertex set $S \subseteq V(G)$* is defined as

$\partial(S) = |B(S)| - |S|$, where $B(S)$ is the set of vertices in $V(G) - S$ that have a neighbor in S, and the *differential of a graph* G is defined as $\partial(G) = \max\{\partial(S) \mid S \subseteq V(G)\}$. A set $S \subseteq V(G)$ is a *∂-set* of G if $\partial(S) = \partial(G)$.

Theorem F. [3] *Let G be a graph.*

(i) *Then $\gamma_R(G) + \partial(G) = |V(G)|$.*
(ii) *An RDF $f = (V_0; V_1; V_2)$ is a γ_R-function on G if and only if V_2 is a ∂-set of G and $V_0 = B(V_2)$.*

3. Upper bounds

Observation 1. *A graph G is in $\mathcal{R}_{UVR}$ if and only if all its components are also in $\mathcal{R}_{UVR}$.*

Observation 2. *Let a graph G be in $\mathcal{R}_{UVR}$. Then G is a Roman graph. If $f = (V_0^f; V_1^f; V_2^f)$ is a γ_R-function on G then $V_1^f = \emptyset$, V_2^f is a γ-set of G and for any $v \in V_2^f$, $|pn[v, V_2^f]| \geq 3$. If D is a γ-set of G then $h = (V(G) - D; \emptyset; D)$ is a γ_R-function on G.*

Proof. As $G \in \mathcal{R}_{UVR}$, it follows by Lemma B that $V_1^g = \emptyset$ for any γ_R-function g on G. Now Theorem D implies a graph G is Roman. Let f be a γ_R-function on G. Since V_2^f is a dominating set of G and $\gamma(G) = \gamma_R(G)/2 = |V_2^f|$, V_2^f is a γ-set of G. Assume $v \in V_2^f$ and $|pn[v, V_2^f]| < 3$. If $v \notin pn[v, V_2^f]$ then $l_1 = ((V_0^f - pn[v, V_2^f]) \cup \{v\}; pn[v, V_2^f]; V_2^f - \{v\})$ is an RDF on G with weight at most $\gamma_R(G)$ and $V_1^g \neq \emptyset$, a contradiction. If $v \in pn[v, V_2^f]$ then $l_2 = (V_0^f - pn[v, V_2^f]; pn[v, V_2^f]; V_2^f - \{v\})$ is an RDF on G with weight not more $\gamma_R(G)$ and $V_1^g \neq \emptyset$, again a contradiction. Thus, $|pn[v, V_2^f]| \geq 3$.

Finally, the weight of h is $2|D| = 2\gamma(G) = \gamma_R(G)$ which shows that h is a γ_R-function on G. □

Remark 3. Let f be a γ_R-function on a graph $G \in \mathcal{R}_{UVR}$. If $v \in V_2^f$ then $|pn[v, V_2^f]|$ can be arbitrarily large. Indeed, let us consider the graph $G = (H_1 \cup H_2) + e$, where H_1 and H_2 are disjoint copies of K_r, $r \geq 4$. Clearly $\gamma_R(G) = 4$, $G \in \mathcal{R}_{UVR}$ and

if $x_i \in V(H_i)$, $i = 1, 2$, then $f = (V(G) - \{x_1, x_2\}; \emptyset; \{x_1, x_2\})$ is a γ_R-function on G and $|pn[x_i, V_2^f]| \in \{r-1, r\}$.

It is easy to see that the following grahs are in $\mathcal{R}_{UVR}$: (a) K_n, $n \geq 3$; (b) $K_{m,n}$ for $m \geq n \geq 4$, (c) P_{3k} and C_{3k}, $k \geq 1$; (d) the cube and icosahedron.

Chambers et al. [4] proved that if G is a graph with $\delta(G) \geq 1$ then $\gamma_R(G) \leq 4n/5$. For the graphs in $\mathcal{R}_{UVR}$ this bound can be lowered.

Proposition 4. *Let $G \in \mathcal{R}_{UVR}$ be a connected graph of order n. Then $\frac{2}{3}n \geq \gamma_R(G)$. If the equality holds then for any γ_R-function f on G, V_2^f is an efficient dominating set of G and each vertex of V_2^f has degree 2. If G has an efficient dominating set D and each vertex of D has degree 2 then $\frac{2}{3}n = \gamma_R(G)$.*

Proof. Let f be any γ_R-function on G. By Observation 2, $V_1^f = \emptyset$ and $|pn[v, V_2^f]| \geq 3$ when $v \in V_2^f$. Hence
(2)
$$|V_0^f| = |\cup_{v \in V_2^f}(N(v) - V_2^f)| \geq \Sigma_{v \in V_2^f}(|pn[v, V_2^f]| - 1) \geq 2|V_2^f| = \gamma_R(G).$$

Therefore, $n = |V_0^f| + |V_2^f| \geq \frac{3}{2}\gamma_R(G)$.

Suppose $n = \frac{3}{2}\gamma_R(G)$. Then all the above inequalities must be equalities. If equality holds on the left side of (2) then $N[v] = pn[v, V_2^f]$ for all $v \in V_2^f$, which implies V_2^f is an efficient dominating set in G. If in addition, the right side of (2) becomes equality then $|N(v)| = 2$ for each $v \in V_2^f$.

Assume now that D is an efficient dominating set of G and all vertices of D have degree 2. Hence $n = 3|D|$. Since G is Roman and each efficient dominating set is a γ-set $\gamma_R(G) = 2\gamma(G) = 2|D| = \frac{2}{3}n$ as required. □

The bound in Proposition 4 is tight at least for all cycles C_{3k}, $k \geq 1$. In the next section we present a constructive characterization of all trees T with $|V(T)| = \frac{3}{2}\gamma_R(T)$.

Jafari Rad and Volkmann in [16] proved that $b_R(G) \leq deg(x) + deg(y) + deg(z) - |N(x) \cap N(y)| - 3$ for any path $P : x, y, z$ in a graph G. For all graphs G belonging to $\mathcal{R}_{UVR}$, this bound can be improved to $\delta(G)$.

Proposition 5. *Let G be a graph and $v \in V(G)$. If for any γ_R-function f on G, $f(v) \neq 1$ then $\gamma_R(G - E_v) > \gamma_R(G)$, where E_v is the set of all edges incident to a vertex v. In particular, $b_R(G) \leq deg(v, G) \leq \Delta(G)$.*

Proof. By Lemma C, $\gamma_R(G - E_v) \geq \gamma_R(G)$. Consider any γ_R-function g on $G - E_v$. Clearly g is an RDF on G. Since v is an isolated vertex in $G - E_v$, $g(v) = 1$. But then g is no γ_R-function on G. Thus $\gamma_R(G - E_v) > \gamma_R(G)$ which implies $b_R(G) \leq deg(v, G) \leq \Delta(G)$. □

Corollary 6. *If a graph G is in $\mathcal{R}_{UVR}$ then $b_R(G) \leq \delta(G)$.*

Proof. By applying Proposition 5 to the graph G and a vertex v of minimum degree we obtain the result. □

The bound stated in Corollary 6 is tight. For example when (a) $G = C_{3k}$, $k \geq 1$, and (b) $\delta(G) = 1$. As an immediate consequence we obtain:

Corollary 7. *For any tree T in $\mathcal{R}_{UVR}$, $b_R(T) = 1$.*

Note that for a tree T of order at least three Ebadi and PushpaLatha [7], and Jafari Rad and Volkmann [16], independently proved that $b_R(T) \leq 3$.

4. Small number of edges

In this section we give necessary and sufficient conditions for a tree to be in $\mathcal{R}_{UVR}$. In particular we present here a constructive characterization of $\mathcal{R}_{UVR}$-trees using labellings. We define a labeling of a tree T as a function $S : V(T) \to \{A, B, C\}$. The label of a vertex v is also called its status, denoted $sta_T(v)$. A labeled tree is denoted by a pair (T, S). We denote the sets of vertices of

status A, B and C by $S_A(T)$, $S_B(T)$ and $S_C(T)$, respectively, or simply by S_A, S_B and S_C if the tree T is clear from context. By a labeled $K_{1,2}$ we shall mean a copy of $K_{1,2}$ whose leaves have status A and the status of the central vertex is B.

Let $\mathscr{T}$ be the family of labeled trees (T, S) that can be obtained from a sequence $(T_1, S_1), \ldots, (T_j, S_j)$, $(j \geq 1)$, of labeled trees such that (T_1, S_1) is a labeled $K_{1,2}$ and $(T, S) = (T_j, S_j)$, and, if $j \geq 2$, (T_{i+1}, S_{i+1}) can be obtained recursively from (T_i, S_i) by one of the four operations $O1$, $O2$, $O3$ and $O4$ listed below.

Operation *O1*. The labeled tree (T_{i+1}, S_{i+1}) is obtained from (T_i, S_i) by adding a path x, y, z and the edge ux where $u \in V(T_i)$ and $sta(u) \in \{A, C\}$, and letting $sta(x) = sta(z) = A$ and $sta(y) = B$.

Operation *O2*. The labeled tree (T_{i+1}, S_{i+1}) is obtained from (T_i, S_i) by adding a star with leaves x, z, t and a central vertex y, and then adding the edge ux where $u \in V(T_i)$ and $sta(u) = B$, and letting $sta(x) = C$, $sta(z) = sta(t) = A$ and $sta(y) = B$.

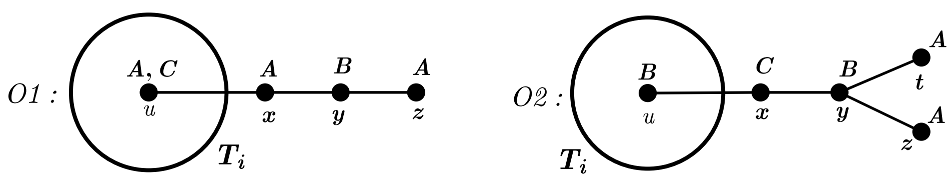

FIGURE 1. Operations $O1$ and $O2$

Operation *O3*. The labeled tree (T_{i+1}, S_{i+1}) is obtained from (T_i, S_i) by adding a path x, y, z and the edge uy where $u \in V(T_i)$ and $sta(u) = C$, and letting $sta(x) = sta(z) = A$ and $sta(y) = B$.

By a labeled tree R we shall mean a labeled tree obtained from a labeled $K_{1,2}$ by operation $O2$.

Operation *O4*. The labeled tree (T_{i+1}, S_{i+1}) is obtained from (T_i, S_i) by adding a labeled R and the edge ux where $u \in V(T_i)$ and $sta(u) \in \{A, C\}$, and $x \in V(R)$ with $sta(x) = C$.

Remark that once a vertex is assigned a status, this status remains unchanged as the labeled tree (T, S) is recursively constructed.

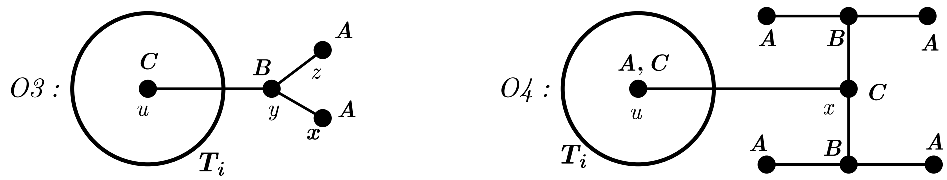

FIGURE 2. Operations $O3$ and $O4$

Observation 8. *Let (T, S) be in $\mathscr{T}$. Then*

(i) *S_B is an independent dominating set and for each vertex v in S_B, $|N(v) \cap S_A| = 2$ and $pn[v, S_B] = (N(v) \cap S_A) \cup \{v\}$.*
(ii) *Each vertex in S_A is adjacent to exactly one vertex in S_B and $|S_A| = 2|S_B|$.*
(iii) *Each vertex in S_C is adjacent to at least 2 vertices in S_B.*
(iv) *S_B is the unique γ-set of T.*

Proof. (i)–(iii) By the definition of (T, S).
(iv) Theorem E and (i) together imply the result. □

Corollary 9. *If $(T, S_1), (T, S_2) \in \mathscr{T}$ then $S_1 \equiv S_2$.*

Proof. Immediately by Observation 8. □

Let $(T, S) \in \mathscr{T}$. By the above corollary, S is unique. So, when the context is clear we shall write $T \in \mathscr{T}$ instead of $(T, S) \in \mathscr{T}$.

We define the following classes of graphs:

- ∂_{CVR} is the class of graphs G such that $\partial(G-v) \neq \partial(G)$ for all $v \in V(G)$, and
- ∂_{UVR} is the class of graphs G such that $\partial(G-v) = \partial(G)$ for all $v \in V(G)$.

By Theorem F it immediately follows the next observation.

Observation 10. *$\mathcal{R}_{UVR} = \partial_{UVR}$ and $\mathcal{R}_{CVR} = \partial_{CVR}$.*

Theorem 11. *For any tree T of order at least three the following assertions are equivalent.*

(i) *T is in $\mathscr{T}$.*
(ii) *T is in $\mathcal{R}_{UVR}$.*

(iii) *T has a unique γ_R-function, say f, and all the following holds: $V_1^f = \emptyset$, V_2^f is independent and $|pn[v, V_2^f]| = 3$ for each $v \in V_2^f$.*

(iv) *T has a unique γ-set D, D is independent and $|pn[v, D]| = 3$ for each $v \in D$.*

(v) *T is in ∂_{UVR}.*

Proof. For any labeled tree $(H, S) \in \mathscr{T}$, $S_B(H)$ is a dominating set of T (by Observation 8) and hence $f_H = (S_A(H) \cup S_C(H); \emptyset; S_B(H))$ is an RDF on H.

Claim 1: If f_H is the unique γ_R-function on $(H, S) \in \mathscr{T}$ then H is in $\mathcal{R}_{UVR}$.

Proof. Since f_H is the unique γ_R-function on $(H, S) \in \mathscr{T}$ and $V_1^{f_H}$ is empty, Lemma B implies that $\gamma_R(H - x) \geq \gamma_R(H)$ for each $x \in V(H)$. If $x \in S_A(H) \cup S_C(H)$ then $f_H|_{H-x}$ is an RDF on $H - x$ of weight $\gamma_R(H)$. Now, let $x \in S_B(H)$. Then x has exactly 2 private neighbors with respect to $S_B(H)$ (by Observation 8) , say y and z. Define $f_H^x : \{0, 1, 2\} \to V(H - x)$ as $f_H^x(y) = f_H^x(z) = 1$ and $f_H^x(t) = f_H(t)$ otherwise. Since the weight of f_H^x is $\gamma_R(H)$, we obtain $H \in \mathcal{R}_{UVR}$.

(i) $\Rightarrow$ (ii): Let (T, S) be in $\mathscr{T}$. By Claim 1, it is sufficient to prove that f_T is actually the unique γ_R-function on T. We now proceed by induction on $|S_B|$. The base case is immediate since T is a labeled star $K_{1,2}$. Let $k \geq 2$ and suppose that for all labeled trees $(H, S') \in \mathscr{T}$ with $|S'_B(H)| < k$ that $f_H = (S'_A(H) \cup S'_C(H); \emptyset; S'_B(H))$ is the unique γ_R-function on H. By Claim 1, $H \in \mathcal{R}_{UVR}$.

Let $(T, S) \in \mathscr{T}$ have $|S_B(T)| = k$. Then T can be obtained from a sequence $T_1 = K_{1,2}, T_2, .., T_k = T$ of labeled trees, and T_{i+1} can be obtained from T_i by operation $O1$, $O2$, $O3$ or $O4$ for $i = 1, .., k-1$. All T_i are clearly in $\mathscr{T}$. We consider four possibilities depending on whether T is obtained from $U = T_{k-1}$ by operation $O1$, $O2$, $O3$ or $O4$. Note that $U \in \mathcal{R}_{UVR}$.

Case 1: T is obtained from $U = T_{k-1}$ by operation $O1$. Suppose T is obtained from U by adding a path x, y, z and the edge ux

where $u \in V(U)$ and $sta(u) \in \{A, C\}$, $sta(x) = sta(z) = A$ and $sta(y) = B$. Clearly $f_T|_U = f_U$ which leads to $\gamma_R(U) = f_U(V(U)) = f_T(V(T)) - 2 \geq \gamma_R(T) - 2$.

Now let f be any γ_R-function on T. Suppose $f(u) \geq 1$. Then the weight of $f|_U$ would be greater than $\gamma_R(U)$ and $f(x)+f(y)+f(z) = 2$, which leads to $\gamma_R(T) > 2+\gamma_R(U)$, a contradiction. So, $f(u) = 0$ for each γ_R-function f on T.

Since $U \in \mathcal{R}_{UVR}$, it follows that $\gamma_R(U - u) = \gamma_R(U)$. Hence if $f(x) \geq 1$ then $f(x) + f(y) + f(z) > 3$ and this implies $f(V(T)) > \gamma_R(U) + 2$, a contradiction. Thus $f(x) = 0$ and then $f(y) = 2$, $f(z) = 0$ and $f|_U = f_U$. All this implies that $f \equiv f_T$ is the unique γ_R-function on T and $\gamma_R(T) = \gamma_R(U) + 2$.

Case 2: T is obtained from $U = T_{k-1}$ by operation $O2$. Suppose T is obtained from U by adding a star $K_{1,3}$ with leaves x, z, t and a central vertex y, and also adding the edge ux where $u \in V(U)$, $sta(u) = sta(y) = B$, $sta(x) = C$ and $sta(z) = sta(t) = A$. Since obviously $f_T|_U = f_U$, $\gamma_R(U) = f_U(V(U)) = f_T(V(T)) - 2 \geq \gamma_R(T) - 2$.

Let f be an arbitrary γ_R-function on T. Hence either $f(y) = 0$ and $f(z) = f(t) = 1$, or $f(y) = 2$ and $f(z) = f(t) = 0$. In the former case we have $f(x) = 2$, which leads to $f(u) = 0$. But then since $U \in \mathcal{R}_{UVR}$, $\gamma_R(U) + 4 \leq f(V(T)) = \gamma_R(T)$, a contradiction. Hence $f(y) = 2$ and $f(z) = f(t) = 0$. But then $f(x) = 0$ and $f|_U = f_U$. From the above we conclude that $f \equiv f_T$ is the unique γ_R-function on T and $\gamma_R(T) = \gamma_R(U) + 2$.

Case 3: T is obtained from $U = T_{k-1}$ by operation $O3$. Suppose T is obtained from U by adding a path x, y, z and the edge uy where $u \in V(T_i)$, $sta(u) = C$, $sta(x) = sta(z) = A$ and $sta(y) = B$. Since $f_T|_U = f_U$, we obtain $\gamma_R(U) = f_U(V(U)) = f_T(V(T)) - 2 \geq \gamma_R(T) - 2$.

Now we shall prove that $f_U|_{U-u}$ is the unique γ_R-function on $U - u$. Since $\gamma_R(U - u) = \gamma_R(U)$ and u has at least two neigbors in $S_B(U) = V_2^{f_U}$, the restriction of f_U on any component of $U-u$ is a γ_R-function. Suppose there is a γ_R-function g on $U - u$ different

from $f_U|_{U-u}$. Then there is at least one component of $U-u$, say U_r, such that $f_U|_{U_r} \not\equiv g_{U_r}$. Define now an RDF h on U as follows: $h(x) = g|_{U_r}(x)$ when $x \in V(U_r)$ and $h(x) = f_U(x)$ otherwise. But then h and f_U have the same weight, a contradiction. Thus, $f_U|_{U-u}$ is the unique γ_R-function on $U-u$.

Now let f be any γ_R-function on T. Obviously $f(y) \neq 1$. Suppose $f(y) = 0$. Then $f(u) = 2$, $f(x) = f(z) = 1$ and $f|_U$ is an RDF on U. Since f_U is the unique γ_R-function of U and $f_U(u) = 0$, an RDF $f|_U$ has weight more than $\gamma_R(U)$ and then $\gamma_R(T) = f(V(T)) > \gamma_R(U) + 2$, a contradiction. Thus $f(y) = 2$ and then $f(x) = f(z) = 0$. If $f(u) = 2$ then as above we again obtain a contradiction. So $f(u) = 0$ and since $f(y) = 2$, it follows that $f|_{U-u}$ is a γ_R-function on $U-u$. But we already know that $f_U|_{U-u}$ is the unique γ_R-function on $U-u$. Therefore $f \equiv f_T$. Since f was chosen arbitrarily, f_T is the unique γ_R-function on T and $\gamma_R(T) = \gamma_R(U) + 2$.

Case 4: T is obtained from $U = T_{k-1}$ by operation $O4$. In this case $T = U \cup R + ux$, where $u \in V(U)$ with $sta(u) \in \{A, C\}$ and x is a central vertex of R, and $sta(x) = C$. Note that R is in $\mathcal{R}_{UVR}$ and $f_R = (S_A(R) \cup S_C(R); \emptyset; S_B(R))$ is the unique γ_R-function on R. Since $f_T|_U \equiv f_U$ and $f_T|_R \equiv f_R$ it follows that $\gamma_R(U) = f_U(V(U)) = f_T(V(T)) - f_R(V(R)) \geq \gamma_R(T) - 4$. Consider any γ_R-function f on T and suppose $f(u) \neq 0$. Then the weight of $f|_U$ is more than $\gamma_R(U)$ and the weight of $f|_R$ is 4. This leads to $\gamma_R(U) > \gamma_R(T) - 4$, a contradiction. Hence $f(u) = 0$. Assume now that $f(x) \neq 0$. Since U is in $\mathcal{R}_{UVR}$, $f|_U(V(U)) \geq \gamma_R(U)$ and $f|_R(V(R)) \geq 4 + f(x) \geq 5$. Hence $f(V(T)) \geq \gamma_R(U) + 5$, a contradiction. Thus $f(x) = 0$. But then $f|_U \equiv f_U$ and $f|_R \equiv f_R$. Thus $f \equiv f_T$ is the unique γ_R-function on T and $\gamma_R(T) = \gamma_R(U) + 4$.

(ii) $\Rightarrow$ (iii): Let a tree T be in $\mathcal{R}_{UVR}$ and let $v \in V_2^f$ for some γ_R-function f on T.

Claim 2. Let u be a neighbor of v and T_u be the component of $T - v$ that contains u. Then

(a) $f|_{T_u}(V(T_u)) \leq \gamma_R(T_u) \leq f|_{T_u}(V(T_u)) + 1$.

(b) There are exactly 2 components of $T - v$, say Q_1 and Q_2, such that $\gamma_R(Q_i) = f|_{T_u}(V(Q_i)) + 1$, $i = 1, 2$.

(c) If $u_i \in V(Q_i)$ is a neighbor of v, $i = 1, 2$, then $pn[v, V_2^f] = \{u_1, u_2, v\}$.

(d) V_2^f is an independent set.

Proof. (a) Assume $\gamma_R(T_u) < f|_{T_u}(V(T_u))$ and let g be any γ_R-function on T_u. But then the function h defined by $h(x) = f(x)$ when $x \in V(T) - V(T_u)$ and $h(x) = g(x)$ when $x \in V(T_u)$, is an RDF on T with weight less than $f(V(T))$, which is impossible.

Now assume $\gamma_R(T_u) > f|_{T_u}(V(T_u))$. Hence $f|_{T_u}$ is no RDF on T_u. Then $f(u) = 0$ and $V_2^{f|T_u}$ dominated $T_u - u$. Define an RDF l on T_u by $l(u) = 1$ and $l(x) = f(x)$ for all $x \in V(T_u - u)$. Since $l(V(T_u)) = f|_{T_u}(V(T_u)) + 1$, the right side inequality is true.

(b) Since $\gamma_R(T - v) = \gamma_R(T)$ and $f(v) = 2$, the result follows by (a).

(c) By the proof of this claim up to here we know that $V_2^f - \{v\}$ dominates $N(v) - \{u_1, u_2\}$ and neither u_1 nor u_2 is dominated by $V_2^f - \{v\}$. Thus $u_1, u_2 \in pn[v, V_2^f]$. If $v \notin pn[v, V_2^f]$ then the RDF g on T defined by $g(x) = f(x)$ for all $x \in V(T) - \{u_1, u_2, v\}$, $g(v) = 0$ and $g(u_1) = g(u_2) = 1$ is a γ_R-function on T with $V_1^g \neq \emptyset$, a contradiction.

(d) The result immediately follows by (c).

We are now ready to prove the uniqueness of f. Suppose there is a vertex $u \in N(v)$ such that $g(u) = 2$ for some γ_R-function g on T. By Claim 2, $g \not\equiv f$ and $g(v) = 0$. Let without loss of generality, $u \not\equiv u_1$. Hence $g|_{Q_1}$ is a γ_R-function on T_1. By the proof of Claim 2 we already know that there is a γ_R-function l on Q_1 with $l(u_1) = 1$. Consider now the γ_R-function g_1 on T defined by $g_1(x) = l(x)$ when $x \in V(Q_1)$ and $g_1(x) = g(x)$ otherwise. Since $g_1(u_1) = l(u_1) = 1$, $V_1^{g_1}$ is not empty, a contradiction.

(iii) $\Rightarrow$ (iv): Since V_2^f is a dominating set of T, by Theorem E it follows that V_2^f is the unique γ-set of T.

(iv) $\Rightarrow$ (i): Denote by $\mathscr{H}$ the set of all trees T for which (iv) holds. We shall prove that if $T \in \mathscr{H}$ then $T \in \mathscr{T}$. We proceed by induction on the domination number of T. If $T \in \mathscr{H}$ and $\gamma(T) = 1$ then $T \equiv K_{1,2}$ and we are done. So, let $T \in \mathscr{H}$, $\gamma(T) = k \geq 2$ and suppose that each tree $H \in \mathscr{H}$ with $\gamma(H) < k$ is in $\mathscr{T}$. Let $P : x_1, x_2, .., x_n$ be any diametral path in T. Then x_n is a leaf and $x_{n-1} \in D$, where D is the unique γ-set of T.

Case 1: $deg(x_{n-1}) = 2$. Since $T \in \mathscr{H}, \{x_{n-2}, x_{n-1}, x_n\} = pn[x_{n-1}, D]$ and all neighbors of x_{n-2} but x_{n-1} are in $V(T) - D$. Now by the choice of P, $N(x_{n-2}) = \{x_{n-1}, x_{n-3}\}$. But then $T - x_{n-3}x_{n-2}$ has exactly 2 components, say F_1 and F_2, where $F_1 \equiv K_{1,2}$ is induced by $\{x_{n-2}, x_{n-1}, x_n\}$. Since the set $D_1 = D - \{x_{n-1}\}$ is an independent dominating set of F_2 and $|pn[v, D_1]| = 3$ for each $v \in D_1$, Theorem E implies that D_1 is the unique γ-set of F_2. Hence $F_2 \in \mathscr{H}$. By inductive hypothesis, $F_2 \in \mathscr{T}$. Since $x_{n-3} \notin D_1$, $sta_{F_2}(x_{n-3}) \in \{A, C\}$ (by Observation 8). Let us consider F_1 as a labeled $K_{1,2}$. But then T is obtained from F_2 by operation $O1$. Thus, $T \in \mathscr{T}$.

Case 2: $deg(x_{n-1}) \geq 3$. By the choice of P, x_{n-2} is the unique non-leaf neighbor of x_{n-1}. Now $deg(x_{n-1}) \geq 3$, $|pn[x_{n-1}, D]| = 3$ and D is independent together imply (a) x_{n-1} is adjacent to exactly 2 leaves, x_n and say y, and (b) $|pn[x_{n-1}, D]| = \{x_{n-1}, x_n, y\}$. First suppose x_{n-2} is adjacent to at least 3 vertices in D. Then $T - x_{n-2}x_{n-1}$ has exactly 2 components, say F_3 and F_4, where $F_3 \equiv K_{1,2}$ with $V(F_3) = \{x_{n-1}, x_n, y\}$. Since the set $D_2 = D - \{x_{n-1}\}$ is an independent dominating set of F_4 and $|pn[v, D_2]| = 3$ for each $v \in D_2$, Theorem E implies that D_2 is the unique γ-set of F_4. Hence $F_4 \in \mathscr{H}$. By inductive hypothesis, $F_4 \in \mathscr{T}$ and by Observation 8, D_2 consists of all vertices having status B. Since $x_{n-2} \notin D_2$ and x_{n-2} is adjacent to at least 2 elements of D_2, again by Observation 8 it follows x_{n-2} has status C. Let us consider F_3 as a member of $\mathscr{T}$. But then T is obtained from F_4 by operation $O3$. Thus, $T \in \mathscr{T}$.

So, let z and x_{n-1} are all neighbors of x_{n-2} in D. Suppose first that $N(x_{n-2}) = \{x_{n-1}, z\}$. Then $T - x_{n-2}z$ has exactly 2 components, say F_5 and F_6, where $V(F_5) = \{x_{n-2}, x_{n-1}, x_n, y\}$. Since the set $D_3 = D - \{x_{n-1}\}$ is an independent dominating set of F_6 and $|pn[v, D_3]| = 3$ for each $v \in D_3$, Theorem E implies that D_3 is the unique γ-set of F_6. Hence $F_6 \in \mathscr{H}$. By inductive hypothesis, $F_6 \in \mathscr{T}$. Define labeling $S : V(T) \Rightarrow \{A, B, C\}$ as follows: (a) the restriction of S on F_6 coincide with the unique labeling of F_6 as a member of $\mathscr{T}$, and (b) $S(x_{n-2}) = C$, $S(x_{n-1}) = B$ and $S(x_n) = S(y) = A$. Since $z \in D_3$, $S(z) = B$ (by Observation 8). But then T is obtained from F_6 by operation $O2$. Thus, $T \in \mathscr{T}$. Finally let x_{n-2} have neighbors in $V(T) - D$. By the choice of P, (a) x_{n-2} has exactly one neighbor in $V(T) - D$, say u, and (b) z is a support vertex of degree 3 which has 2 leaves as neighbors. Then $T - x_{n-2}u$ has exactly 2 components, say F_7 and F_8, where $u \in V(F_8)$ and F_7 is an unlabeled R. Since the set $D_5 = D - \{x_{n-1}, z\}$ is an independent dominating set of F_8 and $|pn[v, D_5]| = 3$ for each $v \in D_5$, Theorem E implies that D_5 is the unique γ-set of F_8. Hence $F_8 \in \mathscr{H}$. By inductive hypothesis, $F_8 \in \mathscr{T}$.

Define labeling $S' : V(T) \Rightarrow \{A, B, C\}$ as follows: (a) the restriction of S' on F_8 coincide with the unique labeling of F_8 as a member of $\mathscr{T}$, and (b) F_7 together with the restriction of S' on F_7 form a labelled R. But then T is obtained from F_8 by operation $O4$. Thus, $T \in \mathscr{T}$.

(ii) $\Leftrightarrow$ (v): Immediately by Observation 10. ■ □

By the proof of the previous theorem it immediately follows:

Corollary 12. *If $(T, S) \in \mathscr{T}$ then $f_T = (S_A(T) \cup S_C(T); \emptyset; S_B(T))$ is the unique γ_R-function on T.*

The class URD of all graphs which have exactly one γ_R-function was introduced and investigated by Chellali and Rad in [5]. Theorem 11 shows that any tree in $\mathcal{R}_{UVR}$ is also in URD.

Corollary 13. *Let f be the unique γ_R-function on a tree $T \in \mathcal{R}_{UVR}$. If $u, v \in pn[x, V_2^f]$ for some $x \in V_2^f$ then $\gamma_R(T - \{u, v\}) = \gamma_R(T) - 1$.*

Proof. By Theorem 11, $|pn[x, V_2^f]| = 3$ and $x \in pn[x, V_2^f]$. Define an RDF g on $T - u$ by $g = (V_0^f - pn[x, V_2^f]; pn[x, V_2^f] - \{u\}; V_2^f - \{x\})$. Since f and g have the same weights, g is a γ_R-function on $T - u$. Now, $v \in V_1^g$ and Lemma B lead to $\gamma_R(T - \{u, v\}) = \gamma_R(T) - 1$. □

Corollary 14. *Let f be the unique γ_R-function on a tree $T \in \mathcal{R}_{UVR}$. If $x, y \in V_0^f$ and $xy \in E(G)$ then $T - xy$ and all its components are in $\mathcal{R}_{UVR}$.*

Proof. Since $x, y \in V_0^f$, $f_1 = f|_{T-xy}$ is an RDF on $T - xy$. Hence $\gamma_R(T - xy) \leq \gamma_R(T)$. Now by Lemma C it follows that $\gamma_R(T - xy) = \gamma_R(T)$. Therefore f_1 is a γ_R-function on $T - xy$ and any γ_R-function on $T - xy$ is a γ_R-function on T. By the uniqueness of f on T it follows that f_1 is the unique γ_R-function on $T - xy$. Since $V_i^{f_1} = V_i^f$ for $i = 1, 2, 3$, the statement (iii) of Theorem 11 holds for any component U of $T - xy$ and $f_1|_U$. Thus U is in $\mathcal{R}_{UVR}$ because of Theorem 11. But then $T - xy$ is also in $\mathcal{R}_{UVR}$ (by Observation 1). □

Recall that for any graph $G \in \mathcal{R}_{UVR}$, $\frac{2}{3}|V(G)| \geq \gamma_R(G)$ (Proposition 4). Now we characterize all trees T for which $\frac{2}{3}|V(T)| = \gamma_R(T)$. Define $\mathscr{T}_1 = \{(T, S) \in \mathscr{T} \mid S_C(T) = \emptyset\}$. Clearly $(T, S) \in \mathscr{T}_1$ if and only if it is sufficient to use only the operation $O1$ for building (T, S) from a labeled $K_{1,2}$

Proposition 15. *If a tree $T \in \mathcal{R}_{UVR}$ then $\frac{2}{3}|V(T)| = \gamma_R(T)$ if and only if $T \in \mathscr{T}_1$.*

Proof. By Theorem 11 and Corollary 12 it follows that $(T, S) \in \mathscr{T}$ and $f_T = (S_A(T) \cup S_C(T); \emptyset; S_B(T))$ is the unique γ_R-function on T. Then $\gamma_R(T) = 2|S_B(T)|$ and $|S_A(T)| = 2|S_B(T)|$ (by Observation 8). Now $|V(G)| = |S_A(T)| + |S_B(T)| + |S_C(T)| \geq$

$|S_A(T)|+|S_B(T)| = \frac{3}{2}\gamma_R(T)$ with equality if and only if $S_C(T) = \emptyset$, as required. □

Theorem 16. *Let a connected n-order graph G be in $\mathcal{R}_{UVR}$. Let the size of G be minimum.*

(i) *Then $|E(G)| = n-1$ if and only if $n \in \{3,6,7\} \cup \{9,10,...\}$ and $G \in \mathscr{T}$.*
(ii) *If $n \in \{4,5\}$ then $|E(G)| = 2n-3$ and $G = K_2 \vee \overline{K_{n-2}}$.*
(iii) *If $n = 8$ then $|E(G)| = 8$ and G is the graph depicted in Figure 3.*

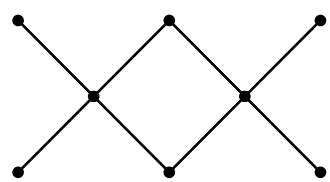

FIGURE 3. The unique 8-vertex unicyclic graph in $\mathcal{R}_{UVR}$.

Proof. (i) If G is a tree then by Theorem 11, G is in $\mathscr{T}$. Now either G is a labeled $K_{1,2}$ or there is a labeled tree $U \in \mathscr{T}$ such that G is obtained from U by applying one of operations $O1$, $O2$, $O3$ and $O4$ once. Hence the order of G is $|V(U)|+3$ or $|V(U)|+4$ or $|V(U)|+7$. This immediately implies the desired result.

(ii) By checking all connected graphs of order 4 and 5 we obtain the result (all such graphs can be found for example in [9] , pages 215–217).

(iii) Let $C_k : x_1, x_2, .., , x_k, x_1$ be the unique cycle in G and f a γ_R-function on G. Since V_2^f is a dominating set of G and $|pn[v, V_2^f]| \geq 3$ for each $v \in V_2^f$ (by Observation 2), $|V_2^f| \leq 2$ and $k \leq 6$. If $|V_2^f| = 1$ then $G = K_{1,7} + e \notin \mathcal{R}_{UVR}$. So, let without loss of generality, $V_2^f = \{x_1, x_i\}$. Clearly neither x_1 nor x_i has more than 2 leaves as neighbors. If $k = 6$ then $i = 4$ and either each of x_1 and x_4 have exactly one leaf as a neighbor or one of x_1 and x_4 is adjacent to 2 leaves. In both cases $G \notin \mathcal{R}_{UVR}$. If $k = 5$ then $i \in \{3,4\}$ and one of x_1 and x_i has 2 leaves as neighbors. But then

$G \notin \mathcal{R}_{UVR}$. Let $k = 4$. Now x_1 and x_i has 2 leaves as neighbors. If $i \in \{2, 4\}$ then $G \notin \mathcal{R}_{UVR}$. Thus $i = 3$ and G is the graph depicted in Figure 3. Clearly $G \in \mathcal{R}_{UVR}$. $\square$

We conclude with three open problems.

Problem 1. *Characterize all unicyclic graphs that are in $\mathcal{R}_{UVR}$.*

Recall that all cycles in $\mathcal{R}_{UVR}$ are C_{3k}, $k \geq 1$.

Problem 2. *For any pair of positive integers n and $k \leq \frac{2}{3}n$ find the maximum integer $s(n, k)$ such that there is an n-order graph $G \in \mathcal{R}_{UVR}$ with $\gamma_R(G) = k$ and $|E(G)| = s(n, k)$.*

Liu and Chang [13] proved that if G is a graph with $\delta(G) \geq 3$ then $\gamma_R(G) \leq 2n/3$. By Proposition 4 we have $\gamma_R(G) < 2n/3$ when $G \in \mathcal{R}_{UVR}$ and $\delta(G) \geq 3$. So, the following problem naturally arises.

Problem 3. *Find an attainable constant upper bound for $\gamma_R(G)/|V(G)|$ on all connected graphs $G \in \mathcal{R}_{UVR}$ with $\delta(G) \geq 3$.*

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